# Teoria i praktyka programowania gier komputerowych

## Linear algebra for games

Piotr Fulmański

Wydział Matematyki i Informatyki,
Uniwersytet Łódzki, Polska

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# Spis treści

# Points and systems of coordinates

In geometry, topology and related branches of mathematics a spatial point is a primitive notion upon which other concepts may be defined. In geometry, points are zero-dimensional; i.e., they do not have volume, area, length, or any other higher-dimensional analogue.
Although there are spaces wher point can be defined. For example, introducing Cartesian coordinates in Euclidean space a point can be defined as an ordered pair, triplet etc. of real numbers.
However: *One way to think of the Euclidean plane is as a set of points satisfying certain relationships*[1].

- Cartesian coordinates
- cylindrical coordinates
- spherical coordinates

[1] In http://en.wikipedia.org/wiki/Euclidean_space

# Left- and right-handed Cartesian coordinate systems

(see. [Mad, 2014], Fig. 3.12)

# Vector

A 3D vector can be represented by a triple of scalars $(x, y, z)$, just as a point can be. The distinction between points and vectors is actually quite subtle. Technically, a vector is just an offset relative to some known point. A vector can be moved anywhere in 3D space – as long as its magnitude and direction don't change, it is the same vector.
As a cosequence we can say that a vector has no concept of position . This means that two vectors are identical as long as they have the same magnitude (or length) and point in the same direction.
A vector can be used to represent a point, provided that we fix the tail of the vector to the origin of the chosen coordinate system. Such a vector is sometimes called a *position vector* or *radius vector*. We can interpret any triple of scalars as either a point or a vector. One might say that points are **absolute**, while vectors are **relative**.

Important: to know graphical inrerpretation (see. [Mad, 2014], Fig. 3.5)

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- If the dot product between two vectors results in 0, it means they are perpendicular to each other (because $\cos(90) = 0$).
- If the dot product results in 1, it means the vectors are parallel and facing in the same direction
- If the dot product results in -1 means they are parallel and face in the opposite direction.
- If $u$ is a unit vector, then the dot product $v \cdot u$ represents the length of the **projection** of a vector $v$ onto the infinie line defined by the direction of $u$. (see. [Mad, 2014], Fig. 3.6 (b))

The cross product between two vectors results in a third vector. Given two vectors, there is only a single plane that contains both vectors. The cross product finds a vector that is perpendicular to this plane, which is known as a normal to the plane (see. [Mad, 2014], Fig. 3.8 (a)).

$$w = u \times v = [(u_y v_z - u_z v_y), (u_z v_x - u_x v_z), (u_x v_y - u_y v_x)]$$

To help memmorize this formula it's good to notice that

- Each component of $w$ is of the form $u_{s_1} v_{s_2} - u_{s_3} v_{s_4}$, with subscripts $s_1, \ldots, s_4$.
- Subscripts $s_1, \ldots, s_4$ for the first component takes values from string `xyzzy`.
- Subscripts for the next component takes values from the above string but with letter substituted in the following manner

$$x \to y \to z \to x$$

The magnitude of the cross product $u \times v$ is equal to the area of the parallelogram whose sides are $u$ and $v$ and is equal to

$$|u \times v| = |u||v| \sin(\theta).$$

- An important thing to note is that there is a second vector that is perpendicular to the plane: the vector that points in the **opposite** direction of $w$ . To find which one is correct we have to use the concept of handedness of the coordinate system: in this case right-handed coordinate system. In a right-handed system, you take your right hand and create a 90 angle between your thumb and index finger. Then make a 90 angle between index finger and middle finger. You then line up your index finger with $u$ and your middle finger with $v$. The direction your thumb points in is the direction the cross product will face. (see [Mad, 2014], Fig. 3.8).

- To get a second vector it's enough to flip fingers so index finger lines up with $v$ and middle finger with $u$.
- $v \times w \neq w \times v$
- $v \times w = -w \times v$
- $v \times (w + y) = (v \times w) + (v \times y)$
- the Cartesian basis vectors are related by cross products as follows

$$\begin{aligned} i \times j &= -(j \times i) &= k \\ j \times k &= -(k \times j) &= i \\ k \times i &= -(i \times k) &= j \end{aligned}$$

- If the cross product returns a vector where all three components are 0, this means that the two input vectors are collinear (they lie on the same line).
- If the dot product results in 1, it means the vectors are parallel and facing in the same direction
- If the dot product results in $-1$ means they are parallel and face in the opposite direction.
- If $u$ is a unit vector, then the dot product $v \cdot u$ represents the length of the **projection** of a vector $v$ onto the infinie line defined by the direction of $u$. (see. [Mad, 2014], Fig. 3.6 (b))

Very often 3D games use 4D vectors. When 4D coordinates are used for a 3D space, they are known as homogenous coordinates , and the fourth component is denoted as the $w$-component .
In most instances, the w-component will be either 0 or 1.

- If $w = 0$, this means that the homogenous coordinate represents a 3D vector.
- If $w = 1$, this means that the homogenous coordinate represents a 3D point.

# Matrix

# Matrix – multiplicaiton

There are three methods (approach) you can think of matrix multiplication.

- From definition.
- Coefficients-vector.
- List of vectors.

# Matrix – multiplicaiton – two forms

There are two methods to represent a vector as a matrix: It could be

- a matrix with a single row (so called row-major) or
- a matrix with a single column (so called column-major).

The choice detemines the form of the matrix we will use in the future and the order of vector and matrix during multiplication.

# Matrix – multiplicaiton for row-major vector

$$\begin{matrix} & & & a & d & g \\ x & y & z & b & e & h \\ & & & c & f & i \end{matrix}$$

# Matrix – multiplicaiton for column-major vector

$$\begin{matrix} a & b & c & x \\ d & e & f & y \\ g & h & i & z \end{matrix}$$

# Matrix – remarks

- If matrices $A$ and $B$ are transformation matrices, then the product $P = AB$ is another transformation matrix that perform **both** of the original transformations. For example, if $A$ is a scale matrix and $B$ is a rotation, the matrix $P$ would both scale and rotate the points or vectors to which it is applied.
- Matrix multiplication is often called **concatenation**.
- $AB \neq BA$
- $A(B + C) = AB + AC$
- $A(BC) = (AB)C$
- $(AB)^T = B^T A^T$
- $(AB)^{-1} = B^{-1} A^{-1}$

# Homogeneous coordinates (Współrzędne jednorodne)

- http://pl.wikipedia.org/wiki/Wsp%C3%B3%C5%82rz%C4%99dne_jednorodne
- http://en.wikipedia.org/wiki/Homogeneous_coordinates