Podstawy grafiki 2D

Podstawowe transformacje

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Spis treści

Rotations

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling

$$\begin{bmatrix} sx & 0 \\ 0 & sy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} sx \cdot x \\ sy \cdot y \end{bmatrix}$$

Shearing

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y \end{bmatrix}$$

General transformation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Inverse matrix for 2x2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Finding the matrix for a transformation – definition of the problem

We have the following problem: given independent vectors u_1 and u_2 and any two vectors v_1 and v_2 , find a linear transformation, in matrix form, that sends u_1 to v_1 and u_2 to v_2 .

Finding the matrix for a transformation – solution of the problem; step 1

Let M be the matrix whose columns are u_1 and u_2. Then

$$T : x \to Mx$$

sends e_1 to u_1 and e_2 to u_2.
Therefore

$$T^{-1} : x \to M^{-1}x$$

sends u_1 to e_1 and u_2 to e_2.

Finding the matrix for a transformation – solution of the problem; step 2

Let K be the matrix whose columns are v_1 and v_2. Then

$$R : x \to Kx$$

sends e_1 to v_1 and e_2 to v_2.

Finding the matrix for a transformation – solution of the problem; step 3

Applying first T^{-1} and then R to vector u_1 we send it to v_1 (via e_1). Smilarly for u_2.

$$R(T^{-1}) : x \to KM^{-1}x$$

Thus, the matrix for the transformation sending the vectors u to the v is just KM^{-1}.

Transformations and coordinate systems

In the special case where we want to go from the usual coordinates on a vector to its coordinates in some coordinate system with basis vectors u_1, u_2, which are

- unit vectors
- and mutually perpendicular,

the transformation matrix is one whose rows are the transposes of u_1 and u_2

Transformations and coordinate systems – example

For example, if

$$u_1 = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

and

$$u_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix},$$

then the vector

$$v = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

expressed in u-coordinates, is

$$\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}.$$

Change of basis (very important!!!)

Any child-space position vector p_C can be transformed into a parent-space position vector p_P as follows

$$p_P = M_{C \to P} p_C$$

where transformation matrix

$$M_{C \to P} = \left[\begin{array}{ccc} i_C & j_C & t_C \end{array} \right]$$

and

- i_C is the unit basis vector along the child space X-axis, expressed **in parent space coordinates**;
- j_C is the unit basis vector along the child space Y-axis, **in parent space**;
- t_C is the translation of the child coordinates system relative **to parent space**.

Change of basis – example

We'll consider transformations of the form

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

If we examine the special case where the upper-left corner is a 2×2 identity matrix, we get

$$\begin{bmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x + c \\ y + f \\ 1 \end{bmatrix}$$

Now it's clear that if we pay attention only to the $x-$ and $y-$coordinates, this is a translation.